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General form of the SU(3) Gürsey matrix

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Abstract. The general form of the unitary unimodular Gürsey matrix at the SU(3) level is found in terms of a general hermitian octet vector. The relevance to chiral lagrangian theories of hadron physics is discussed, and other applications are suggested.

1. Introduction

The widespread interest (Gasiorowicz and Geffen 1969), over the last few years, in the chiral algebra of $\mathrm{SU(3)} \times \mathrm{SU(3)}$ has focused attention on the problem of parametrizing a general unitary unimodular three by three matrix in terms of the components of a hermitian octet vector. We shall briefly review the developments in high energy physics which led to this specific problem, before giving a general solution and indicating some of its more immediate uses.

One of the major techniques in studying the application of chiral algebras (both $\mathrm{K(2)} = \mathrm{SU(2)} \times \mathrm{SU(2)}$ and $\mathrm{K(3)} = \mathrm{SU(3)} \times \mathrm{SU(3)}$) to hadronic dynamics has been the construction of explicit chiral invariant Lagrangians with their associated currents (Gasiorowicz and Geffen 1969), and with the subsequent addition of symmetry breaking terms. In order to construct such Lagrangians it is necessary to study the transformation of particle fields under the chiral algebra, and in particular to find the explicit form of transformation law for those fields forming nonlinear realizations of the algebra. The general theory of such realizations is well established (Coleman *et al* 1969 and Isham 1969) but leads directly only to a power series in fields which is difficult to compute beyond low orders. For the case of the K(2) algebra there are two approaches which lead to general closed form expressions for the transformation laws. One of these is the algebraic approach of Weinberg (1968) and the other the matrix method associated with the name of Gürsey (Chang and Gürsey 1967). Both these methods are in principle capable of generalization to the K(3) level but, despite extensive efforts, no general solution has been found. Macfarlane *et al* (1970, and references therein) have shown how the algebraic method leads to either equations of the sixth degree or linked partial differential equations and have found some simple particular solutions, while in the alternative approach they were able to discover three particular models. Although these authors have not found a general solution they have presented by far the most complete treatment of the problem to date, and we recommend it strongly to the interested reader (Macfarlane *et al* 1970, and references therein). For our present purposes we will only indicate most briefly how the unitary matrix enters in the Gürsey approach.

If we consider a multiplet N of spin $\frac{1}{2}$ fields which transform under SU(n) according to the fundamental (quark) representation, so that

$$N \to \exp(\mathrm{i}\theta^i T^i) N \tag{1}$$

where the T^i are the $n \times n$ matrix representations of the algebra, then by imposing

$$N \rightarrow \exp(\mathrm{i}\phi^i T^i \gamma_5) N \tag{2}$$

for the parity changing transformations we have a linear representation of the full K(n) algebra. Here $\gamma_0 \gamma_5^\dagger \gamma_0 = -\gamma_5$, so that the Lagrangian

$$\mathscr{L} = \mathrm{i}\bar{N}\eth N + m\bar{N}N \tag{3}$$

although SU(n) invariant has a mass term which breaks the full chiral invariance. Gürsey's method consists of introducing at this point a unitary matrix function $\hat{U}$ of the octet of hermitian pseudoscalar meson fields M^i which represent the Goldstone bosons of the scheme (Gasiorowicz and Geffen 1969). By choosing the transformation properties of $\hat{U}$ so that $\bar{N}\hat{U}N$ is a full chiral invariant, it is then possible to define a new quark field $\psi = \hat{U}^{1/2}N$ which transforms (nonlinearly in the meson fields) so that an invariant mass term is allowed. If we now write

$$\hat{U} = S + \mathrm{i}P\gamma_5 \tag{4}$$

$$= (S+\mathrm{i}P)\frac{1+\gamma_5}{2} + (S-\mathrm{i}P)\frac{1-\gamma_5}{2} \tag{5}$$

$$= U\frac{1+\gamma_5}{2} + U^\dagger\,\frac{1-\gamma_5}{2} \tag{6}$$

where S and P are respectively scalar and pseudoscalar hermitian $n \times n$ matrix functions of M^i, then it is simple to show that U is also unitary. Now the matrices $\frac{1}{2}(1 \pm \gamma_5)$ are well known to be projection operators so that any required function of $\hat{U}$ may be formed immediately if the same function of U has been constructed, and moreover the transformation laws of $\hat{U}$ and U are similarly trivially related. In order that the term $\bar{N}\hat{U}N$ be invariant $\hat{U}$ must have the transformation law

$$\hat{U} \rightarrow \exp(-\mathrm{i}\phi^i T^i \gamma_5)\hat{U}\,\exp(-\mathrm{i}\phi^j T^j \gamma_5) \tag{7}$$

under chiral action, and the useful infinitesimal result

$$U \rightarrow U - \mathrm{i}\phi^i\{T^i, U\} \tag{8}$$

follows at once. If U is known as an explicit function of the M^i, then the transformation law for the M^i is straightforward to deduce. Such a treatment may be found in Macfarlane's work (Macfarlane *et al* 1970) (for particular choices of the structure of U), where also the generalization required to define the transformation properties of all other fields, in terms of the unitary square root of U, is given. For the present work we content ourselves with this brief description, since the problem is now well posed. We have to construct the most general unitary 3×3 matrix out of the hermitian components M^i of an octet vector, and we must be able to construct simple functions (eg $U^{1/2}$) from it.

2. Parametrization of the matrix

First let us treat the problem at the SU(2) level where the results are known and the machinery familiar, but using an approach which lends itself to generalization. In this case the M^i are simply the pion isovector triplet π^i, there is only one independent isotopic invariant $\pi^2 = \pi^i\pi^i$, and the product law

$$\tau_i\tau_j = \delta_{ij} + i\epsilon_{ijk}\tau_k \tag{9}$$

specifies the properties of the hermitian traceless Pauli matrices. The vector π^i defines only one direction in its space (all directions being equivalent), and if a normalized vector n^i with

$$n^in^i = 1 \tag{10}$$

is taken along this direction, then the fields π^i are specified by the two angles inherent in n^i and by the magnitude π^2. Using this vector n_i we may construct

$$n^{\pm} = \tfrac{1}{2}(1 \pm n_i\tau_i) \tag{11}$$

which, since $n_i\tau_i$ is clearly a matrix square root of unity, are two hermitian projection operators. Now to construct the most general U expand it in the form

$$U = a^+n^+ + a^-n^- \tag{12}$$

where $a^{\pm}$ are functions of π^2, so that the unitarity implies that $a^{\pm}$ are of modulus unity while the determinant of U is a^+a^- which may be absorbed into an overall phase. Therefore, restricting ourselves to unimodular U, we may take

$$U = n_+ \exp(i\theta) + n_- \exp(-i\theta) \tag{13}$$

$$= \exp(i\theta n_i\tau_i) \tag{14}$$

$$= \cos\theta + in_i\tau_i \sin\theta \tag{15}$$

where θ is any hermitian function of π^2, as the general form. If we define

$$f = (\pi^2)^{1/2} \cot\theta \tag{16}$$

then contact with the algebraic work of Weinberg (1968) is immediate. Alternatively, taking

$$a^+ = \frac{1+ib}{1-ib} \tag{17}$$

where b is a hermitian function of π^2, we find the Cayley or rational representation of Macfarlane *et al* (1970), and the connection between this and the exponential treatment is obvious. All such choices of representation are general and equivalent (Weinberg 1968); they contain one arbitrary function of the sole SU(2) invariant π^2. Notice that the expansion (13) of U in terms of projection operators allows functions of U to be formed trivially and, in particular, the unitary unimodular $U^{1/2}$ found by halving the 'angle' θ.

As stated above, the advantage of this treatment is that it leads to a fairly straightforward extension at the SU(3) level. This time the hermitian traceless matrices satisfy the product law

$$\lambda_i\lambda_j = \tfrac{2}{3}\delta_{ij} + d_{ijk}\lambda_k + if_{ijk}\lambda_k \tag{18}$$

the M^i form an octet vector of SU(3), and there are two independent SU(3) invariants

$$X = M^iM^i \tag{19}$$

and

$$Y = d_{ijk}M^iM^jM^k \tag{20}$$

in general. The vector M^i defines two directions in its space since

$$N^i = d_{ijk}M_jM_k \tag{21}$$

with

$$3N_iN_i = X^2 \tag{22}$$

is in general an independent vector. These vectors have been extensively studied by Macfarlane *et al* (1970, and references therein) and by Michel and Radicati (1968), and we now quote several of their results without giving any proof. First they show that

$$f_{ijk}M_jN_k = 0 \tag{23}$$

so we see that $M_i\lambda_i$ and $N_i\lambda_i$ are commuting matrices, and we shall speak of the vectors commuting. Next they establish that

$$3d_{ijk}M_jN_k = XM_i \tag{24}$$

and

$$3d_{ijk}N_jN_k = 2YM_i - XN_i \tag{25}$$

and hence the products of all contractions of the λ matrices with the general vectors formed linearly from M_i and N_i can be computed. That such a general vector shall have a positive norm implies

$$3Y^2 \leqslant X^3 \tag{26}$$

and we set

$$Y = \left(\frac{X^3}{3}\right)^{1/2} \sin\phi \tag{27}$$

in the remainder of this work. Finally, Michel and Radicati (1968) study particular vectors which will become relevant for our work at a later stage. They call charge vectors those M^i for which $|Y|$ attains its maximum value, and N^i is parallel to M^i. Special vectors are those M^i for which Y is zero, and N^i (which is therefore orthogonal to M^i) is itself a charge vector. We shall use the symbols q^i and s^i for charge and special vectors of unit norm.

With the above results in mind, we define the vectors m_i and r_i by

$$m_i = X^{-1/2}M_i \tag{28}$$

and

$$r_i = \frac{\sqrt{3}}{X\cos\phi}\left\{N_i - \left(\frac{X}{3}\right)^{1/2} M_i \sin\phi\right\} \tag{29}$$

so that we have

$$m_im_i = 1 = r_ir_i \tag{30}$$

and

$$m_ir_i = 0 \tag{31}$$

as obvious consequences. Now the results given above allow us to compute

$$\sqrt{3}\, d_{ijk}m_jm_k = m_i\sin\phi + r_i\cos\phi = -\sqrt{3}\, d_{ijk}r_jr_k \tag{32}$$

and

$$\sqrt{3}\, d_{ijk} m_j r_k = m_i \cos\phi - r_i \sin\phi \tag{33}$$

by straightforward algebra. Hence, if we define

$$\alpha_i = r_i \cos\alpha + m_i \sin\alpha \tag{34}$$

$$\beta_i = -r_i \sin\alpha + m_i \cos\alpha \tag{35}$$

we easily obtain

$$-\sqrt{3}\, d_{ijk}\alpha_j\alpha_k = \alpha_i \cos(\phi - 3\alpha) + \beta_i \sin(\phi - 3\alpha) = \sqrt{3}\, d_{ijk}\beta_j\beta_k \tag{36}$$

and

$$\sqrt{3}\, d_{ijk}\alpha_j\beta_k = \beta_i \cos(\phi - 3\alpha) - \alpha_i \sin(\phi - 3\alpha) \tag{37}$$

while the relations

$$\alpha_i\alpha_i = 1 = \beta_i\beta_i \tag{38}$$

$$\alpha_i\beta_i = 0 \tag{39}$$

are clearly also valid. Hence, if we demand that β_i is special (and α_i the corresponding charge) we have

$$\sin(\phi - 3\alpha) = 0 \tag{40}$$

$$\cos(\phi - 3\alpha) = 1 \tag{41}$$

and hence

$$\alpha = \tfrac{1}{3}(\phi + 2k\pi) \tag{42}$$

gives solutions when k takes on integer values. Of course there are only two independent vectors at all stages and we shall set k equal to zero, thus using one special vector and its associated charge. With this choice we now have

$$-\sqrt{3}\, d_{ijk} q_j q_k = q_i = \sqrt{3}\, d_{ijk} s_j s_k \tag{43}$$

$$\sqrt{3}\, d_{ijk} s_j q_k = s_i \tag{44}$$

where the signs have been taken so as to compare directly with the use of the third and eighth directions as respectively special and charge directions in 'The Eightfold Way' (Gell-Mann and Ne'eman 1964). This choice of sign was made in equation (41), and the alternate choice corresponds merely to reversing the sign of all components of the vectors.

With this machinery at our disposal we now proceed by analogy with the SU(2) case. Since a general vector M^i now gives us in reality two commuting vectors, we expect three projection operators related to the (3×3) matrix cube roots of unity. The properties of the projection operators may be summarized by

$$P_\alpha P_\beta = 0 = P_\beta P_\alpha \qquad (\alpha \neq \beta; \alpha = 0, +, -) \tag{45}$$

$$(P_\alpha)^2 = P_\alpha \tag{46}$$

$$P_0 + P_+ + P_- = 1 \tag{47}$$

where 1 is the unit matrix, and these projection operators are hermitian with unit trace. Now the matrices

$$\omega^r P_0 + \omega^s P_+ + \omega^t P_- \tag{48}$$

where r, s and t are 0, 1 or 2, and $\omega = \exp(\frac{2}{3}i\pi)$ is a complex cube root of unity, are clearly 27 matrix cube roots of 1. Obviously r can be set equal to zero by extracting an algebraic factor, and from the resulting nine matrices we select the three with unit determinant

$$1 = P_0 + P_+ + P_- \tag{49}$$

$$N_+ = P_0 + \omega P_+ + \omega^2 P_- \tag{50}$$

$$N_- = P_0 + \omega^2 P_+ + \omega P_- \tag{51}$$

as our basic cube roots. The latter two of these roots are hermitian adjoints of each other, have zero trace (so that we might hope to construct them from λ^i matrices), and obey

$$N_+ N_+ = N_- \qquad N_- N_- = N_+ \tag{52}$$

$$N_+ N_- = 1 = N_- N_+ \tag{53}$$

in agreement with corresponding equations on the algebraic cube roots. Immediately we can invert the equations (49), (50) and (51) into the form

$$3P_0 = 1 + N_+ + N_- \tag{54}$$

$$3P_+ = 1 + \omega^2 N_+ + \omega N_- \tag{55}$$

$$3P_- = 1 + \omega N_+ + \omega^2 N_- \tag{56}$$

where the identity

$$1 + \omega + \omega^2 = 0 \tag{57}$$

is used for the algebraic roots.

To obtain the detailed relationship between the charge vectors and these cube roots, we write

$$N_+ = N^i_+ \lambda_i = (a_i - ib_i)\lambda_i \tag{58}$$

$$N_- = N^i_- \lambda_i = (a_i + ib_i)\lambda_i \tag{59}$$

where a_i and b_i are hermitian. Substituting into (52), and using (53), the relations

$$a_i a_i = \tfrac{3}{4} = b_i b_i \tag{60}$$

$$a_i b_i = 0 \tag{61}$$

$$f_{ijk} a_j b_k = 0 \tag{62}$$

$$a_k = 2d_{ijk} a_i a_j = -2d_{ijk} b_i b_j \tag{63}$$

$$b_i = -2d_{ijk} a_j b_k \tag{64}$$

are easily derived. We identify b_i as a special vector with a_i as the corresponding charge and, with the normalization

$$2a_i = -\sqrt{3}\, q_i \tag{65}$$

$$2b_i = \sqrt{3}\, s_i \tag{66}$$

make contact with our earlier results.

The machinery is now complete, and the lesson is clearly that, rather than work with the M^i, one should use X, α, and the six independent components in $N^i_{\pm}$, while employing the P_i wherever the matrix form is desired. In particular the relations

$$M^i = \left(\frac{X}{3}\right)^{1/2} [N_i^+ \exp\{i(\alpha+\tfrac{1}{2}\pi)\} + N_i^- \exp\{-i(\alpha-\tfrac{1}{2}\pi)\}] \tag{67}$$

and

$$M = M^i\lambda_i = -\left(\frac{4X}{2}\right)^{1/2} \left\{P_0 \sin\alpha + P^+ \sin\left(\alpha+\frac{2\pi}{3}\right) + P_- \sin\left(\alpha-\frac{2\pi}{3}\right)\right\} \tag{68}$$

are especially useful, and follow immediately from our previous results. Now, to specify the most general unitary unimodular U in terms of the M^i, we write

$$U = U_0P_0 + U_+P_+ + U_-P_- = U_\alpha P_\alpha \tag{69}$$

where the U_α are functions of the SU(3) invariants X and Y (or ϕ), and the P_α are specified from the M^i as above. The unitary condition yields

$$|U_0|^2 = |U_+|^2 = |U_-|^2 = 1 \tag{70}$$

while

$$U_0U_+U_- = 1 \tag{71}$$

follows from unimodularity. Thus the situation is entirely analogous to that at the SU(2) level, and we simply present the exponential version as an example of the infinitely many equivalent general representations. We write

$$U = \exp i(\theta N^+ + \theta^* N^-) \tag{72}$$

$$= P_0 \exp i(\theta+\theta^*) + P_+ \exp i(\omega\theta + \omega^2\theta^*) + P_- \exp i(\omega^2\theta + \omega\theta^*) \tag{73}$$

where θ is any complex algebraic function of X and Y, and where the relation $1+\omega+\omega^2 = 0$ has been used to maintain maximum symmetry. It should be noted here that the P_α are given above explicitly so that U is expressed directly in terms of M^i, the λ^j matrices, and the two arbitrary hermitian functions of X and Y inherent in θ. Moreover, because the P_α are projection operators, the unitary unimodular $U^{1/2}$ is trivial to find in any representation; here it is obtained merely by halving the 'angles' θ. Notice that, if M^i is a special vector then $Y = 0$, so that θ and θ^* are functions of X which is now the sole SU(3) invariant. If M^i is a charge vector then only one direction is specified and $|Y|$ attains its maximum value. If θ is taken to be hermitian, for example, then the special vector defined by $(N_+ - N_-)$ is not specified. Hence, in the charge case, only two projection operators (P_0 and $(P_+ + P_-)$) are defined and the latter has trace two. With these modifications the general formalism applies, and the results may be read off directly even in these singular cases.

3. Conclusions

The immediate and most important application of these results is, of course, to the specification of nonlinear realizations of K(3) (Dondi 1971, pp 100–19 and Sarkar 1971, pp 95–140) and the construction of the corresponding chiral Lagrangian (Sarkar 1971, pp 95–140) as described above. However, if we set

$$\theta = \left(\frac{X}{3}\right)^{1/2} \exp\{\mathrm{i}(\alpha + \tfrac{1}{2}\pi)\} \tag{74}$$

then

$$U = \exp(\mathrm{i}M_i\lambda_i) \tag{75}$$

so that viewing the M^i now as simply a set of eight real parameters we have produced a concrete expression for a finite SU(3) transformation on the fundamental (quark) representation as a by-product of our result. This would seem to lend itself rather naturally to use in problems of the type recently suggested by Dashen (1971), where finite (rather than infinitesimal) transformations are crucial. Recently, Rosen (1971) also has produced concrete expressions for such finite SU(3) transformations. Although his work is earlier than ours we were not aware of its existence when we performed our calculations, and it is therefore encouraging that his results indeed appear as a subset of ours when the limit (74) is applied. The only advantage in this particular application is that our general case directly includes the special and charge cases (as may be checked by setting $\alpha = 0$ or $\pm\frac{1}{6}\pi$), but this improvement may prove to be a 'critical' one.

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